Reachability Problems for Continuous Linear Dynamical Systems

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(Joint work with Ventsislav Chonev and Joël Ouaknine)

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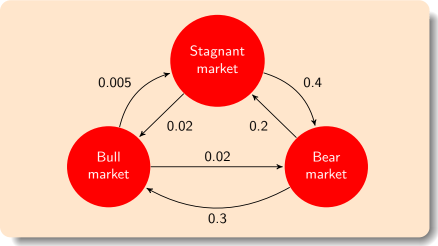
Reachability for Continuous-Time Markov Chains



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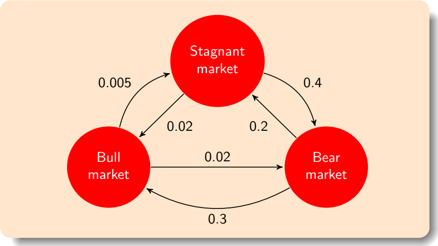

Distribution $P(t)$ at time t satisfies $P'(t) = P(t)Q$, where

$$Q = \begin{pmatrix} -0.025 & 0.02 & 0.005 \\ 0.3 & -0.5 & 0.2 \\ 0.02 & 0.4 & -0.42 \end{pmatrix}$$

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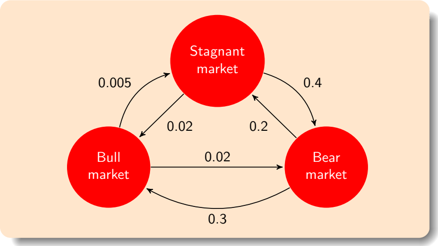

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- Require that π not be on boundary of the target set.

Cyber-Physical Systems

"To analyze a cyber-physical system, such as a pacemaker, we need to consider the **discrete software controller** *interacting with the physical world, which is typically modelled by* **differential equations***"*

Rajeev Alur (CACM, 2013)

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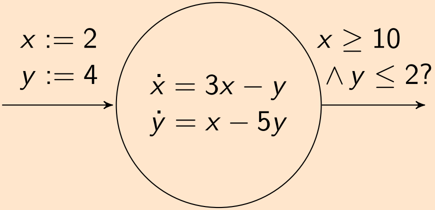

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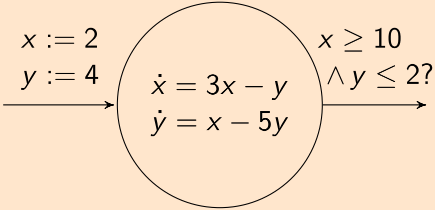

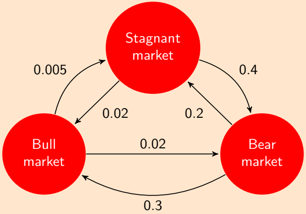

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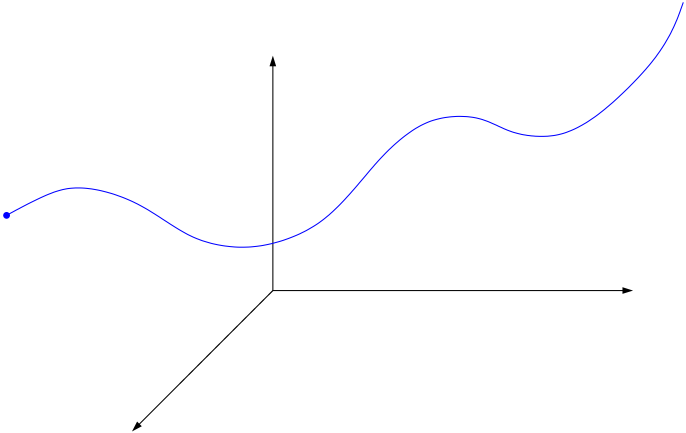

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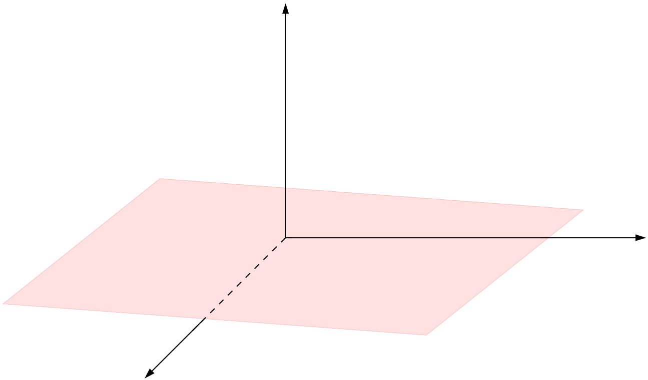

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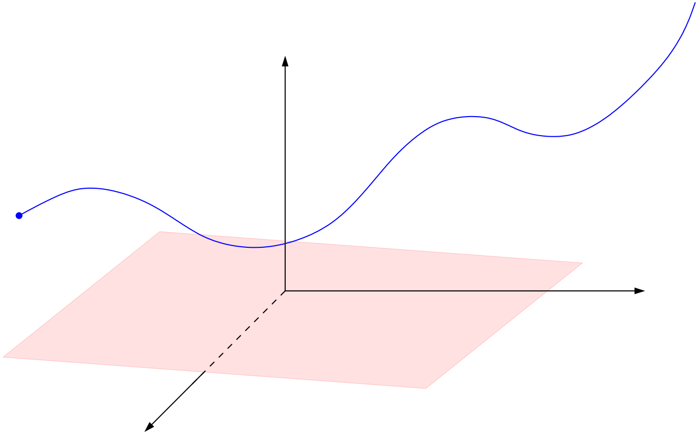

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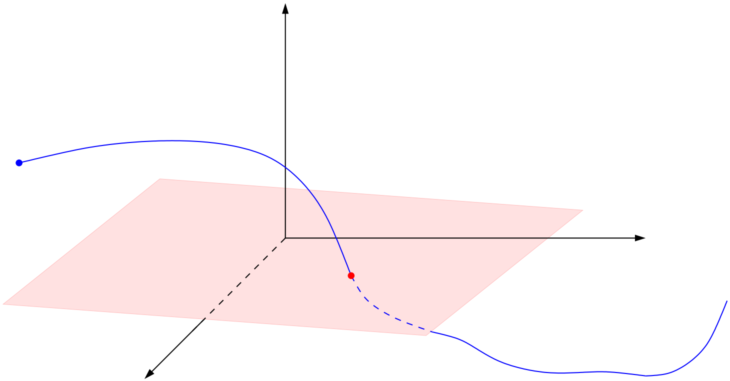

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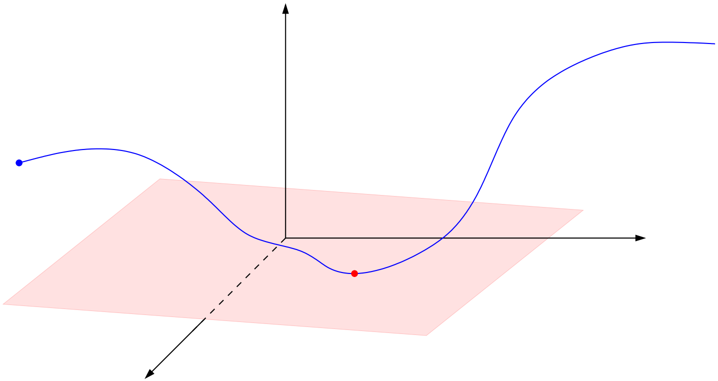

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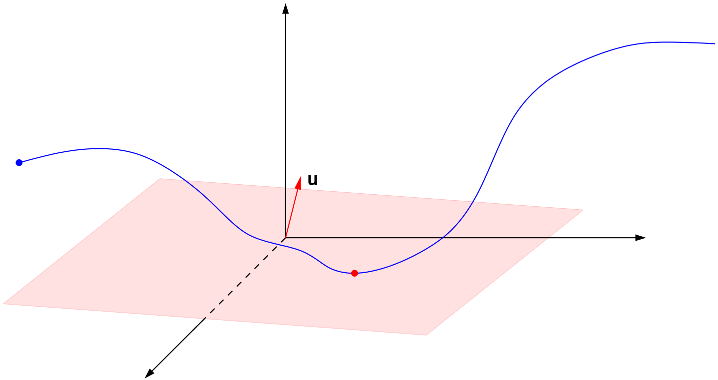

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Note – the λ_j are complex in general.

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Instance: f and bounded interval $[a, b]$
Question: Is there $t \in [a, b]$ such that $f(t) = 0$?

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- **Decidability open!** [Bell, Delvenne, Jungers, Blondel 2010]

Related Work

A lot of work since 1920s on the zeros of exponential polynomials

$$f(z) = \sum_{j=1}^{m} P_j(z) e^{\lambda_j z}$$

(Polya, Ritt, Tamarkin, Kac, Voorhoeve, van der Poorten, ...)
but mostly on distribution of *complex* zeros.

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CONTINUOUS-ORBIT Problem

The problem of whether the trajectory $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$ reaches a given **target point** was shown to be decidable by Hainry (2008) and in PTIME by Chen, Han and Yu (2015).

Our Results

Theorem (Chonev, Ouaknine, W. 2015)

Assuming Schanuel's Conjecture, BOUNDED-ZERO is decidable at all orders.

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At order 9, if ZERO is decidable then the Diophantine approximation type of any real algebraic number α is a computable number.

It turns out that decidability in the bounded case follows from a much more general result, discovered (but not published) in the early 1990s by Macintyre and Wilkie.

[Angus Macintyre, personal communication, July 2015]

Schanuel's Conjecture

Theorem (Lindemann-Weierstrass)

If $a_1, \ldots, a_n$ are algebraic numbers linearly independent over $\mathbb{Q}$, then $e^{a_1}, \ldots, e^{a_n}$ are algebraically independent.

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Easy Consequence

By Schanuel's conjecture, some two-element subset of $\{1, \pi i, e^1, e^{\pi i}\}$ is algebraically independent.

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Theorem (Macintyre and Wilkie 1996)

The first-order theory of $(\mathbb{R}, +, \cdot, e^x)$ is decidable, assuming Schanuel's conjecture.

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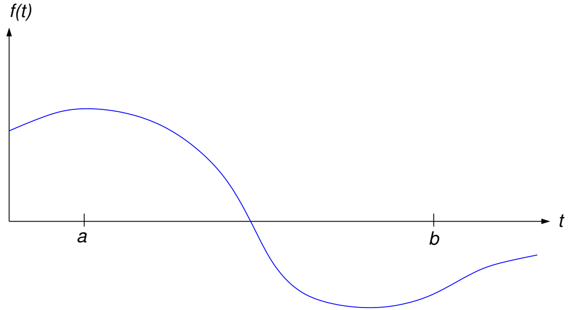

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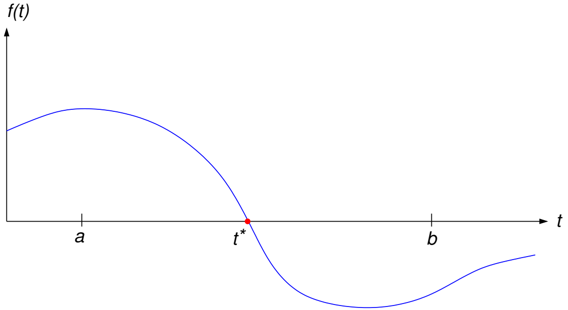

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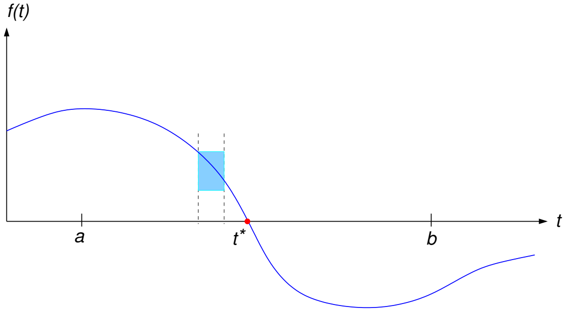

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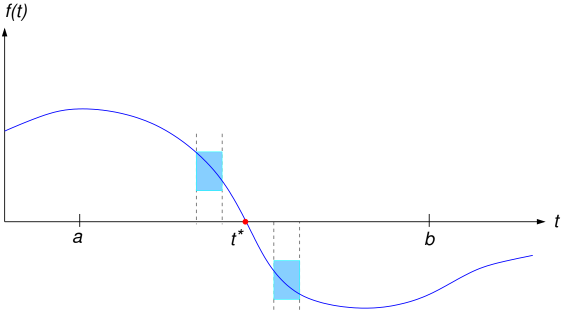

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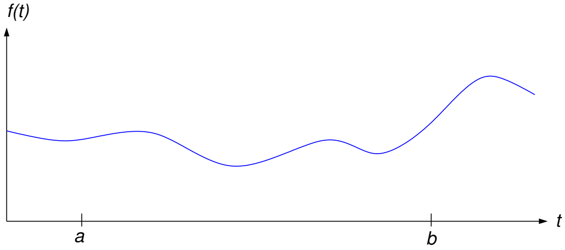

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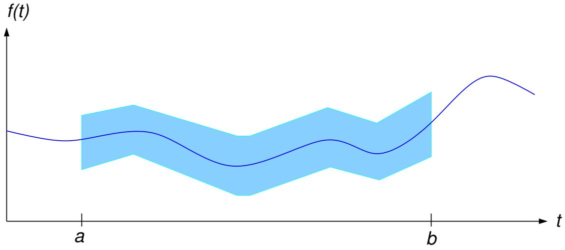

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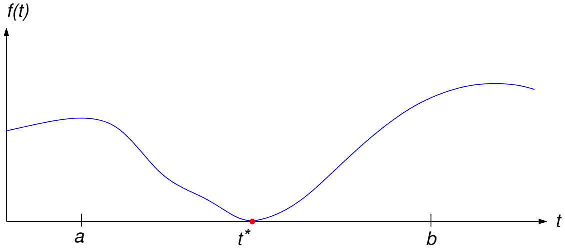

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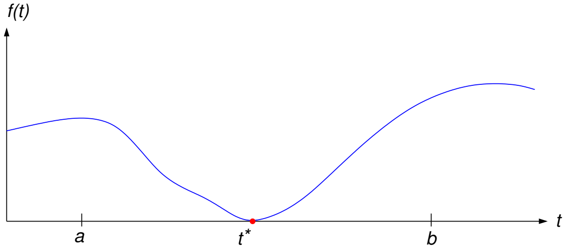

Can this situation arise?

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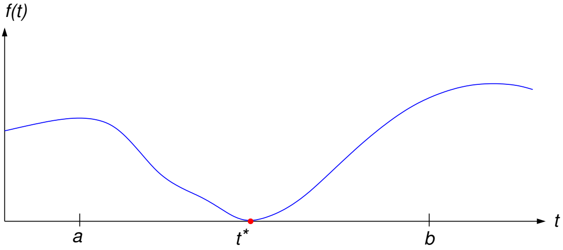

Easily! For example, $f(t) = 2 + e^{it} + e^{-it}$.

Laurent Polynomials and Factorisation

Example

- Write $f(t) = 2 + e^{it} + e^{-it}$ in the form $f(t) = P(e^{it})$ for the **Laurent polynomial**

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Idea: factorise f. Noting that factors may be complex-valued!

Laurent Polynomials and Factorisation

Any exponential polynomial $f(t)$ can be written

$$f(t) = P(t, e^{a_1 t}, \ldots, e^{a_m t})$$

with

$$P \in \mathbb{C}[x, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$$

and $\{a_1, \ldots, a_m\}$ a set of complex algebraic numbers linearly independent over $\mathbb{Q}$.

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Proof Strategy

Factorisation of P into irreducible factors induces factorisation of f. Assuming Schanuel's conjecture, we can decide the existence of zeros of real-valued and complex-valued irreducible factors.

The Unbounded Case

ZERO Problem

Instance: f

Question: Is there $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$?

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How well can one approximate a real number x with rationals?

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Theorem (Dirichlet 1842)

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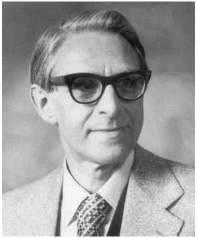
Theorem (Roth 1955)

Let $x \in \mathbb{R}$ be algebraic. Then for any $\varepsilon > 0$ there are finitely many integers p, q such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}.$$



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Definition

Let $x \in \mathbb{R}$. The **Diophantine-approximation type** $L(x)$ is:

$$L(x) = \inf \left\{ c : \left| x - \frac{p}{q} \right| < \frac{c}{q^2} \text{ for some } p, q \in \mathbb{Z} \right\}.$$

Continued Fractions

Finite continued fractions:

$$[3, 7, 15, 1, 292] = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}$$

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Infinite continued fractions:

$$[a_0, a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

Real Algebraic Numbers

Theorem

The continued fraction expansion of a real quadratic irrational number is periodic.

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$$\sqrt{389} = [19, 1, 2, 1, 1, 1, 1, 2, 1, 38, 1, 2, 1, 1, 1, 1, 2, 1, 38, \ldots]$$

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What about numbers of degree ≥ 3?

$$\sqrt[3]{2} = [1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, 1 \\ 3, 4, 1, 1, 2, 14, 3, 12, 1, 15, 3, 1, 4, 534, 1, 1, 5, 1, 1, \ldots]$$

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Lang and Trotter: "*no significant departure from random behaviour*"

An Open Problem

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"Is there an algebraic number of degree higher than two whose simple continued fraction has unbounded partial quotients? Does every such number have unbounded partial quotients?"

R. K. Guy, 2004

A Mathematical Obstacle at Order 9

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Theorem (Chonev, Ouaknine, W., 2015)

If the ZERO PROBLEM is decidable at order 9 then there is an algorithm that given a real algebraic number α computes $L(\alpha)$ to arbitrary precision. In particular, the set

$$\{\alpha \in \overline{\mathbb{Q}} : \alpha \text{ has bounded partial quotients}\}$$

would be recursively enumerable.

The ZERO Problem at Low Orders

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Instance: f

Question: Is there $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$?

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Diophantine-approximation bounds play a key role in the proof—specifically Baker's theorem on linear forms in logarithms of algebraic numbers.

Illustrative Example

Consider the exponential polynomial

$$f(t) = 2 + \cos(t + \varphi_1) + \cos(\sqrt{2}t + \varphi_2) - e^{-t}$$

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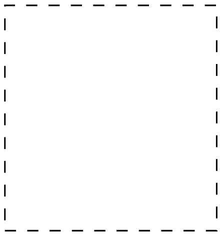

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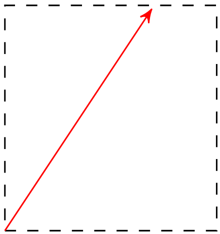

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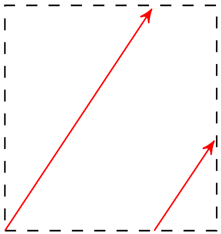

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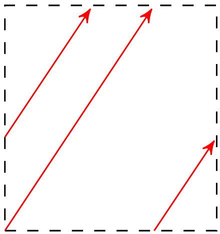

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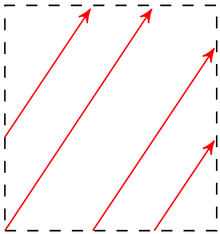

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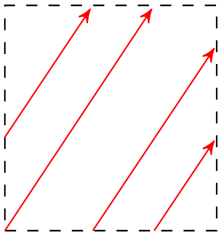

Baker's Theorem:

$$\left|\left|\left(t + \varphi_1, \sqrt{2}t + \varphi_2\right) - (\pi, \pi)\right|\right| \geq \frac{1}{\mathrm{poly}(t)}$$

Conclusion and Perspectives

The Discrete Case

A **linear recurrence sequence** is a sequence $\langle u_0, u_1, u_2, \ldots \rangle$ of integers such that there exist constants $a_1, \ldots, a_k$, such that

$$u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \ldots + a_k u_n$$

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Theorem (Skolem 1934; Mahler 1935, 1956; Lech 1953)

The set of zeros of a linear recurrence sequence is semi-linear:

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Theorem (Berstel and Mignotte 1976)

In Skolem-Mahler-Lech, the infinite part (arithmetic progressions $A_1, \ldots, A_\ell$) is fully constructive.

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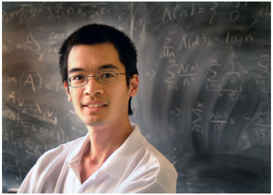

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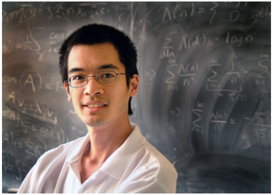

"... a mathematical embarrassment ..."

Richard Lipton

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- Formidable "mathematical obstacle" at dimension 9 in the unbounded case.
- Similar obstacles for the Infinite-Zeros Problem.
- Diophantine-approximation techniques unavoidable.